Simultaneously Orthogonal Expansion of Two Stationary Gaussian Processes — Examples

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This paper presents two examples of the simultaneously orthogonal expansion of the sample functions of a pair of stationary Gaussian processes. The pair of Gaussian processes are specified by zero means and covariances exp $(-\alpha \mid s - t \mid)$, *exp* $(-\beta \mid s - t \mid)$ *in Example 1 and by* $1 - \mid s - t \mid /2T$, *exp* $(- \mid s - t \mid /T)$ *in Example 2. The expansion takes the form of a trigonometric series where the coefficients are mutually independent Gaussian variables for both processes, and the series converges, both with probability one for every t and in the stochastic mean uniformly in t, for both processes. This type of expansion is an extension of the Karhunen-Loéve expansion to the case of a pair of processes, and no concrete example has been given previously.*

The general theory of the orthogonal expansions is briefly reviewed in Section I, while concrete results for the two examples are tabulated in Section II with a brief outline of the method of derivation. The complete derivation, which constitutes the principal part of this paper, is presented in full detail in Appendices.

I. GENERAL THEORY

Orthogonal expansion of Gaussian processes has been used extensively for both theoretical investigation and application in communication engineering. In the case of a single process, the expansion is a modified version of the Karhunen-Loéve expansion.[1,2] Specifically, if $x(t)$, $-T \leqq t \leqq T$, is the sample function of a Gaussian process with zero mean and a continuous covariance $R(s,t)$, $-T \leqq s, t \leqq T$, then $x(t)$ can be expanded in terms of the (orthonormalized) eigenfunctions f_k, $k = 0, 1, 2,$

$\cdots$, of R as follows:*

$$x(t) = \sum_k \xi_k(x) f_k(t), \qquad \text{a.e. } [P],$$

$$\xi_k(x) = (x, f_k), \qquad \text{a.e. } [P],\dagger$$

where P is the Gaussian measure induced by the process with zero mean and the covariance R.

This expansion has two desirable properties: (*i*) for every measurable set specified by the sample function $x(t)$, there is an equivalent measurable set specified by the coefficients, $\{\xi_k\}$, and (*ii*) $\{\xi_k\}$ are mutually independent Gaussian variables with zero means and variances equal to the eigenvalues of R. Thus, many problems concerning the sample function can often be reduced to equivalent problems concerning the coefficients alone, and they in turn can be decomposed into a collection of effectively "one-dimensional problems".

In the case of two Gaussian processes, the sample functions can be expanded relative to a single set of functions such that their coefficients are mutually independent with respect to the two finite dimensional distributions.‡ Such "simultaneously orthogonal" expansions have often been used to prove the equivalence-singularity dichotomy of two Gaussian measures.[3,4,5] The particular expansion theorem used here is due to Pitcher.[5] It goes as follows: Let P_1 and P_2 be the probability measures induced by two Gaussian processes with zero means and continuous, positive-definite covariances $R_1(s,t)$ and $R_2(s,t)$, $-T \leqq s, t \leqq T$.§ If $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ is densely defined and bounded on $\mathfrak{L}_2$ and its extension to the

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* R denotes both the covariance and the integral operator generated by it, namely,

$$(Rf)(t) = \int_{-T}^{T} R(s,t) f(s) ds, \qquad f \in \mathfrak{L}_2,$$

where $\mathfrak{L}_2$ is the space of square-integrable functions on $[-T,T]$.

† Without loss of generality, the process under consideration is assumed to be separable and measurable. (f,g) denotes the usual scalar product of two elements f and g in $\mathfrak{L}_2$.

‡ Instead of regarding "two Gaussian processes" as two *one-parameter families* of random variables, we consider a *single* one-parameter family of measurable functions x_t, $-T \leqq t \leqq T$, with *two* probability measures P_1 and P_2.

§ We assume that P_1 and P_2 are extended to $\mathfrak{B}$ and complete on $\mathfrak{B}_{P_1}$ and $\mathfrak{B}_{P_2}$, where $\mathfrak{B}$ is the minimal σ-field with respect to which x_t is measurable for every $t \in [-T,T]$. Since $R_1(s,t)$ and $R_2(s,t)$ are both continuous, we consider only the separable and measurable version of $\{x_t, -T \leqq t \leqq T\}$ without loss of generality, where the separability is with respect to $\frac{1}{2}(P_1 + P_2)$ while the measurability is with respect to $\mathfrak{B}_{\frac{1}{2}(P_1+P_2)} \times \mathfrak{A}$ and $\mathfrak{A}$ is the Lebesgue field of the subsets of $[-T,T]$. Note that such a version is also separable with respect to both P_1 and P_2 and measurable with respect to $\mathfrak{B}_{P_1} \times \mathfrak{A}$ and $\mathfrak{B}_{P_2} \times \mathfrak{A}$.

whole of $\mathfrak{L}_2$ has a set of eigenfunctions which spans $\mathfrak{L}_2$, then

$$x(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{k=0}^{n} \eta_k(x)(R_1^{\frac{1}{2}}\varphi_k)(t), \qquad [P_1 \times \mu, P_2 \times \mu],$$

$$\eta_k(x) = \underset{j\to\infty}{\text{l.i.m.}}\ (x, R_1^{-\frac{1}{2}}\varphi_{kj}), \qquad [P_1,P_2],$$

where φ_k, $k = 0, 1, 2, \cdots$, are the orthonormalized eigenfunctions of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$, and μ is the Lebesgue measure defined on the subsets of $[-T,T]$, and φ_{kj}, $j = 0, 1, 2, \cdots$, are a sequence of $\mathfrak{L}_2$-functions in the domain of $R^{-\frac{1}{2}}$ that converges strongly to φ_k for each k, namely,

$$\varphi_{kj} \in \mathfrak{D}(R_1^{-\frac{1}{2}}), \qquad \lim_{j\to\infty} \| \varphi_k - \varphi_{kj} \| = 0, \qquad k = 0, 1, 2, \cdots.$$

As in the case of a single process, this expansion also has two desirable properties:

(*i*) for every measurable* set Λ specified by $x(t)$, there exists another measurable set Λ' specified by $\{\eta_k\}$ such that

$$P_1(\Lambda \,\Delta\, \Lambda') = 0 = P_2(\Lambda \,\Delta\, \Lambda'),\dagger$$

$$(ii)\ E_1\{\eta_k\eta_j\} = \delta_{kj}, \qquad E_2\{\eta_k\eta_j\} = \lambda_k\delta_{kj}, \qquad k, j = 0, 1, 2, \cdots,$$

where λ_k is the eigenvalue of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ corresponding to φ_k. As seen from the definition and the property (*ii*), η_k, $k = 0, 1, 2, \cdots$, are, with respect to both P_1 and P_2, mutually independent Gaussian variables with zero means, and their variances are all unity with respect to P_1 and λ_k with respect to P_2.

Unfortunately, the above theorem is not a suitable method of actually obtaining the simultaneous expansion for a given pair of covariances $R_1(s,t)$ and $R_2(s,t)$. As the result of defining the expansion coefficients η_k and the expanding functions $R_1^{\frac{1}{2}}\varphi_k$ in terms of φ_k, $k = 0, 1, \cdots$, one must first of all solve the homogeneous equation involving the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$. Yet, there is no standard method of solution available for this type of equation, since $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ is not, in general, a simple operator as R_1 and R_2 are. This may partly account for the fact that no concrete example for the simultaneous expansion has been given previously. In the next section and Appendices, we give two examples to illustrate an indirect method of obtaining φ_k first and then calculating η_k and $R_1^{\frac{1}{2}}\varphi_k$.

* The measurability is with respect to $\mathfrak{B}_{\frac{1}{2}(P_1+P_2)}$.

† Δ denotes symmetric difference, namely, $\Lambda \,\Delta\, \Lambda' = (\Lambda - \Lambda') \cup (\Lambda' - \Lambda)$.

II. EXAMPLES

The two examples we consider here are stationary Gaussian processes with zero means and the pair of covariances

$$R_1(s,t) = \exp(-\alpha \mid s - t \mid), \qquad R_2(s,t) = \exp(-\beta \mid s - t \mid)$$

in Example 1, and

$$R_1(s,t) = 1 - \frac{\mid s - t \mid}{2T}, \qquad R_2(s,t) = \exp\left(-\frac{\mid s - t \mid}{T}\right)$$

in Example 2.

The method we employ to obtain the expansions may be outlined as follows: First, we prove that $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ is densely defined and bounded, by showing that $R_2^{\frac{1}{2}}R_1^{-\frac{1}{2}}$ is bounded. Next, we consider solutions of the homogeneous equation

$$R_2\psi_k = \lambda_k R_1 \psi_k .$$

If ψ_k is a square-integrable solution with a real number λ_k, then $R_1^{\frac{1}{2}}\psi_k$ is seen to be an eigenfunction of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$. Thus, the function $R_1^{\frac{1}{2}}\varphi_k$ in the desired expansion is simply $R_1\psi_k$.* Unfortunately, there are no such solutions in Example 1, and there are only half of what is needed in Example 2. However, suppose we consider "formal solutions" of the homogeneous equation and expand them relative to the set of eigenfunctions of R_1, which forms an orthonormal basis of $\mathfrak{L}_2$. Let ψ_{kj} be the sum of the first j terms of such an expansion. Then, it turns out that the normalized version of $\{R_1^{\frac{1}{2}}\psi_{kj}\}_j$ is the desired sequence $\{\varphi_{kj}\}$ used for defining η_k. That is, we show that (*i*) the normalized version of $\{R_1^{\frac{1}{2}}\psi_{kj}\}_j$ forms a Cauchy sequence and its limit in the mean is an eigenfunction of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ with λ_k as the corresponding eigenvalue for each k, (*ii*) the collection of all such limits forms an orthonormal basis of $\mathfrak{L}_2$, hence they are the only eigenfunctions of the extension. Thus, the desired expansion is obtained simply by calculating $R_1^{\frac{1}{2}}\varphi_k$, $k = 0, 1, 2, \cdots$.

Finally, that the expansion series in both examples converge both a.e. $[P_1, P_2]$ for every t and in the mean $[P_1, P_2]$ uniformly in t, is deduced as follows: We first observe that the t-functions of the series are uniformly bounded in i and t, and the sum of the variances of the coefficients is finite with respect to both P_1 and P_2. Thus, by virtue of mutual independence of the coefficients, the partial sum of the series, denoted by

* The expansion in terms of $R_1\psi_k$ is suggested in Ref. 6 without the connection with $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$.

$x_n(t)$, converges both a.e. $[P_1, P_2]$ for every t and in the mean $[P_1, P_2]$ uniformly in t to some Gaussian variable $\tilde{x}(t)$, $E_1\{\tilde{x}(t)\} = 0 = E_2\{\tilde{x}(t)\}$. Now, since $x_n(t)$ converges to $x(t)$ in the mean $[P_1 \times \mu, P_2 \times \mu]$, $E_1\{x_n(s)x_n(t)\}$ and $E_2\{x_n(s)x_n(t)\}$ converge in the mean $[\mu \times \mu]$ to $R_1(s,t)$ and $R_2(s,t)$ which are continuous. Hence, they converge uniformly to $R_1(s,t)$ and $R_2(s,t)$, respectively, which implies that

$$E_1\{\tilde{x}(s)\tilde{x}(t)\} = R_1(s,t) \qquad \text{and} \qquad E_2\{\tilde{x}(s)\tilde{x}(t)\} = R_2(s,t).$$

That is, $\{\tilde{x}(t), -T \leqq t \leqq T\}$ is a Gaussian process with zero mean and covariances $R_1(s,t)$ and $R_2(s,t)$ corresponding to P_1 and P_2, respectively. Hence, $\tilde{x}(t) = x(t)$, a.e. $[P_1, P_2]$ for every t. Thus, it follows that $x_n(t)$ converges to $x(t)$ both a.e. $[P_1, P_2]$ for every t and in the mean $[P_1, P_2]$ uniformly in t.

The results are tabulated in the following summary. In both examples, the orthogonal expansions of the sample function take the form of trigonometric series, which asymptotically behave like harmonic series for large indices. The coefficients are mutually independent Gaussian variables with zero means, and their variances are explicitly given, together with the asymptotic values for large indices. In addition, we include for future reference the eigenvalues and the (orthonormalized) eigenfunctions of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ to the whole of $\mathfrak{L}_2$, though they are not of the primary interest here.

III. SUMMARY

3.1 *Example 1*

$$R_1(s,t) = \exp(-\alpha \mid s - t \mid), \qquad R_2(s,t) = \exp(-\beta \mid s - t \mid), \qquad \alpha > \beta > 0.$$

$$x(t) = \sum_{i=0}^{\infty} [\eta_i(x) \cos \theta_i t + \hat{\eta}_i(x) \sin \hat{\theta}_i t], \qquad \text{a.e. } [P_1, P_2],$$

where θ_i's and $\hat{\theta}_i$'s are positive solutions of*

$$(\alpha + \beta)\theta_i \tan \theta_i T = \alpha\beta - \theta_i^2,$$

$$-(\alpha + \beta)\hat{\theta}_i \operatorname{ctn} \hat{\theta}_i T = \alpha\beta - \hat{\theta}_i^2,$$

and $\eta_i, \hat{\eta}_i, i = 0, 1, 2, \cdots$, are mutually independent Gaussian variables with zero means and variances given by

* θ_i's and $\hat{\theta}_i$'s are indexed in the ascending order.

$$E_1\{\eta_i^2\} = 2\alpha/(\alpha^2 + \theta_i^2)\tau(\theta_i), \qquad E_1\{\hat{\eta}_i^2\} = 2\alpha/(\alpha^2 + \hat{\theta}_i^2)\tau(\hat{\theta}_i),$$

$$E_2\{\eta_i^2\} = 2\beta/(\beta^2 + \theta_i^2)\tau(\theta_i), \qquad E_2\{\hat{\eta}_i^2\} = 2\beta/(\beta^2 + \hat{\theta}_i^2)\tau(\hat{\theta}_i),$$

where

$$\tau(\theta) = T + \frac{(\alpha + \beta)\alpha\beta}{\theta^4 + (\alpha^2 + \beta^2)\theta^2 + \alpha^2\beta^2}.$$

For large i,

$$\theta_i \sim \left(i + \frac{1}{2}\right)\frac{\pi}{T}, \qquad \hat{\theta}_i \sim i\frac{\pi}{T},$$

$$E_1\{\eta_i^2\} \sim \frac{2\alpha T}{\left(i + \frac{1}{2}\right)^2 \pi^2}, \qquad E_i\{\hat{\eta}_i^2\} \sim \frac{2\alpha T}{i^2\pi^2},$$

$$E_2\{\eta_i^2\} \sim \frac{2\beta T}{\left(i + \frac{1}{2}\right)^2 \pi^2}, \qquad E_2\{\hat{\eta}_i^2\} \sim \frac{2\beta T}{i^2\pi^2}.$$

The extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ has eigenvalues

$$\lambda_{2i} = \frac{\beta}{\alpha}\frac{\alpha^2 + \theta_i^2}{\beta^2 + \theta_i^2}, \qquad \lambda_{2i+1} = \frac{\beta}{\alpha}\frac{\alpha^2 + \hat{\theta}_i^2}{\beta^2 + \theta_i^2},$$

and orthonormalized eigenfunctions

$$\varphi_{2i}(t) = \frac{2\alpha}{\alpha + \beta}\left[\frac{\alpha^2 + \theta_i^2}{\tau(\theta_i)}\right]^{\frac{1}{2}} \cos\theta_i T \operatorname*{l.i.m.}_{n\to\infty} \sum_{j=0}^{n} \gamma(\sigma_j, \theta_i)\cos\alpha\sigma_j T \cos\alpha\sigma_j t,$$

$$\varphi_{2i+1}(t) = \frac{2\alpha}{\alpha + \beta}\left[\frac{\alpha^2 + \hat{\theta}_i^2}{\tau(\hat{\theta}_i)}\right]^{\frac{1}{2}} \sin\hat{\theta}_i T \operatorname*{l.i.m.}_{n\to\infty} \sum_{j=0}^{n} \gamma(\hat{\sigma}_j, \hat{\theta}_i)\sin\alpha\hat{\sigma}_j T \sin\alpha\hat{\sigma}_j t,$$

where

$$\gamma(\sigma,\theta) = (1 + \sigma^2)^{\frac{1}{2}} \Big/ \left[T + \frac{\sigma^2}{\alpha(1 + \sigma^2)}\right](\alpha^2\sigma^2 - \theta^2),$$

and σ_j and $\hat{\sigma}_j$ are positive solutions of*

$$\sigma_j \tan\alpha\sigma_j T = 1, \qquad -\hat{\sigma}_j \operatorname{ctn}\alpha\hat{\sigma}_j T = 1.$$

* σ_i's and $\hat{}_i$'s are indexed in the ascending order.

3.2 *Example 2*

$$R_1(s,t) = 1 - \frac{|s-t|}{2T}, \qquad R_2(s,t) = \exp\left(-\frac{|s-t|}{T}\right).$$

$$x(t) = \sum_{i=0}^{\infty} [\eta_i(x) \cos \theta_i t + \hat{\eta}_i(x) \sin \theta_i t], \qquad \text{a.e. } [P_1, P_2]$$

where θ_i's are positive solutions of*

$$\theta_i T \tan \theta_i T = 1,$$

and η_i, $\hat{\eta}_i$, $i = 0, 1, 2, \cdots$, are mutually independent Gaussian variables with zero means and variances given by

$$E_1\{\eta_i^2\} = E_1\{\hat{\eta}_i^2\} = \frac{1 + \theta_i^2 T^2}{\theta_i^2 T^2 (2 + \theta_i^2 T^2)},$$

$$E_2\{\eta_i^2\} = E_2\{\hat{\eta}_i^2\} = \frac{2}{2 + \theta_i^2 T^2}.$$

For large i,

$$\theta_i \sim i\frac{\pi}{T}, \qquad E_1\{\eta_i^2\} \sim \frac{1}{i^2\pi^2}, \qquad E_2\{\eta_i^2\} \sim \frac{2}{i^2\pi^2}.$$

The extension of $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ has eigenvalues

$$\lambda_{2i} = \lambda_{2i+1} = \frac{2\theta_i^2 T^2}{1 + \theta_i^2 T^2},$$

and orthonormalized eigenfunctions

$$\varphi_{2i}(t) = \left(T\frac{2 + \theta_i^2 T^2}{1 + \theta_i^2 T^2}\right)^{-\frac{1}{2}} \cos \theta_i t$$

$$\varphi_{2i+1}(t) = 2\varphi_{2i}(T) \operatorname*{l.i.m.}_{n\to\infty} \sum_{j=0}^{n} \frac{(-1)^j (j + \frac{1}{2})\pi}{\theta_i^2 T^2 - (j + \frac{1}{2})^2 \pi^2} \sin\left(j + \tfrac{1}{2}\right)\frac{\pi}{T} t.$$

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* θ_i's are indexed in the ascending order.

APPENDIX A

Example 1

$$R_1(s,t) = \exp(-\alpha\,|\,s - t\,|\,), \qquad -T \leqq s,t \leqq T, \qquad \alpha > \beta > 0.$$
$$R_2(s,t) = \exp(-\beta\,|\,s - t\,|\,),$$

A.1 *Eigenvalues and Eigenfunctions of R_1 and R_2*

Let μ_k and f_k, $k = 0, 1, 2, \cdots$, be the eigenvalues and the corresponding orthonormalized eigenfunctions of R_1. Then,

$$\mu_{2i} = \frac{2}{\alpha(1 + \sigma_i^2)}, \qquad \mu_{2i+1} = \frac{2}{\alpha(1 + \hat{\sigma}_i^2)}, \qquad i = 0, 1, 2, \cdots, \tag{1}$$

$$\begin{aligned} f_{2i}(t) &= \cos \alpha\sigma_i t \Big/ \left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}\right)^{\frac{1}{2}}, \\ f_{2i+1}(t) &= \sin \alpha\hat{\sigma}_i t \Big/ \left(T - \frac{\sin 2\alpha\hat{\sigma}_i T}{2\alpha\hat{\sigma}_i}\right)^{\frac{1}{2}}, \end{aligned} \tag{2}$$

where σ_i and $\hat{\sigma}_i$ are positive solutions of

$$\sigma_i \tan \alpha\sigma_i T = 1, \qquad -\hat{\sigma}_i \operatorname{ctn} \alpha\hat{\sigma}_i T = 1, \tag{3}$$

respectively, indexed in ascending order.*

Similarly, the eigenvalues and eigenfunctions of R_2, denoted by ν_k and g_k, are given by

$$\nu_{2i} = \frac{2}{\beta(1 + \rho_i^2)}, \qquad \nu_{2i+1} = \frac{2}{\beta(1 + \hat{\rho}_i^2)}, \tag{4}$$

$$\begin{aligned} g_{2i}(t) &= \cos \alpha\rho_i t \Big/ \left(T + \frac{\sin 2\beta\rho_i T}{2\beta\rho_i}\right)^{\frac{1}{2}}, \\ g_{2i+1}(t) &= \sin \beta\hat{\rho}_i t \Big/ \left(T - \frac{\sin 2\beta\hat{\rho}_i T}{2\beta\hat{\rho}_i}\right)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where ρ_i and $\hat{\rho}_i$ are positive solutions of

$$\rho_i \tan \beta\rho_i T = 1, \qquad -\hat{\rho}_i \operatorname{ctn} \beta\hat{\rho}_i T = 1. \tag{6}$$

A.2 *Boundedness of $R_2^{\frac{1}{2}}R_1^{-\frac{1}{2}}$*

Since $\{f_k\}$ forms an orthonormal basis of $\mathfrak{L}_2[-T,T]$, we only have to show that $\|\, R_2^{\frac{1}{2}}R_1^{-\frac{1}{2}}f_k \,\|$ is uniformly bounded relative to the index k.

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* See Ref. 7, pp. 99–101.

Namely, we must show existence of a constant c, $0 < c < \infty$, independent of k such that

$$\| R_2^{\frac{1}{2}} R_1^{-\frac{1}{2}} f_k \|^2 = \sum_l \frac{\nu_l}{\mu_k} (f_k , g_l)^2 < c.$$

Consider even k's, and put $k = 2i$, $i = 0, 1, 2, \cdots$. Observe

$$(f_{2i} , g_l) = 0, \qquad l\text{: odd.} \tag{7}$$

Hence, we shall consider only even l's. Note

$$(f_{2i} , g_{2j})^2 = \frac{\left(\int_{-T}^{T} \cos \alpha\sigma_i t \cos \beta\rho_j t \, dt \right)^2}{\left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i} \right) \left(T + \frac{\sin 2\beta\rho_i T}{2\beta\rho_i} \right)}, \qquad j = 0, 1, 2, \cdots,$$

$$\int_{-T}^{T} \cos \alpha\sigma_i t \cos \beta\rho_j t \, dt$$

$$= 2(\alpha\sigma_i \tan \alpha\sigma_i T - \beta\rho_j \tan \beta\rho_j T) \frac{\cos \alpha\sigma_i T \cos \beta\rho_j T}{\alpha^2\sigma_i^2 - \beta^2\rho_j^2} \tag{8}$$

$$= 2(\alpha - \beta) \frac{\cos \alpha\sigma_i T \cos \beta\rho_j T}{\alpha^2\sigma_i^2 - \beta^2\rho_j^2}$$

where (3) and (6) are used for the last equality, and

$$(1 + \sigma_i^2) \cos^2 \alpha\sigma_i T = \sigma_i^2, \qquad (1 + \rho_j^2) \cos^2 \beta\rho_j T = \rho_j^2, \tag{9}$$

which also follow from (3) and (6). Thus,

$$\frac{\nu_{2i}}{\mu_{2j}} (f_{2i} , g_{2j})^2 = \frac{4 \frac{\beta}{\alpha} (\alpha - \beta)^2 \cos^2 \beta\rho_j T}{T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}} \frac{a(\beta\rho_j) \alpha^2 \sigma_i^2}{\beta^2\rho_j^2(\alpha^2\sigma_i^2 - \beta^2\rho_j^2)^2}, \tag{10}$$

where

$$a(\theta) = \frac{\cos^2 \theta T}{T + \frac{\sin 2\theta T}{2\theta}}.$$

Also note that

$$\frac{\alpha^2\sigma_i^2}{\beta^2\rho_j^2(\alpha^2\sigma_i^2 - \beta^2\rho_j^2)^2} = \frac{1}{(\alpha^2\sigma_i^2 - \beta^2\rho_j^2)^2} + \frac{1}{\alpha^2\sigma_i^2\beta^2\rho_j^2}$$
$$+ \frac{1}{2\alpha^3\sigma_i^3} \left(\frac{1}{\alpha\sigma_i + \beta\rho_j} + \frac{1}{\alpha\sigma_i - \beta\rho_j} \right).$$

First, through direct calculation witb the use of (5) and (8),

$$\left[\int_{-T}^{T} \cos \alpha\sigma_i t\, g_{2j}(t)\, dt\right]^2 = 4(\alpha - \beta)^2 \cos^2 \alpha\sigma_i T \frac{a(\beta\rho_j)}{(\alpha^2\sigma_i^2 - \beta^2\rho_j^2)^2},$$

and from the fact tbat $\{g_k\}$ forms an orthonormal basis of $\mathfrak{L}_2$,

$$\sum_{j=0}^{\infty} \left[\int_{-T}^{T} \cos \alpha\sigma_i t\, g_{2j}(t)\, dt\right]^2 = \int_{-T}^{T} \cos^2 \alpha\sigma_i t\, dt$$

$$= T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}.$$

Thus,

$$\sum_{j=0}^{\infty} \frac{4\frac{\beta}{\alpha}(\alpha - \beta)^2 \cos^2 \beta\rho_j T}{\left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}\right)} \frac{a(\beta\rho_j)}{(\alpha^2\sigma_i^2 - \beta^2\rho_j^2)^2} \leqq \frac{\beta/\alpha}{\cos^2 \alpha\sigma_0 T}. \tag{11}$$

Secondly, it follows from (3) and (6) that

$$\begin{aligned} T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i} &= T\left[1 + \frac{1}{\alpha T(1 + \sigma_i^2)}\right] > T, \\ T + \frac{\sin 2\beta\rho_j T}{2\beta\rho_j} &> T, \end{aligned} \tag{12}$$

and

$$\alpha^2\sigma_j^2 > \left(\frac{\pi}{T} j\right)^2, \quad \beta^2\rho_j^2 > \left(\frac{\pi}{T} j\right)^2, \quad j = 1, 2, \cdots. \tag{13}$$

Thus, $a(\beta\rho_j) < 1/T$. Hence,

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{4\frac{\beta}{\alpha}(\alpha - \beta)^2 \cos^2 \beta\rho_j T}{\left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}\right)} \frac{a(\beta\rho_j)}{\alpha^2\sigma_i^2\beta^2\rho_j^2} \\ < \frac{4\frac{\beta}{\alpha}(\alpha - \beta)^2}{\alpha^2\sigma_0^2}\left[\frac{1}{\beta^2\rho_0^2T^2} + \frac{1}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\right]. \end{aligned} \tag{14}$$

Thirdly, let $h_1(z)$ be a function of the complex variable z defined by

$$h_1(z) = \frac{z}{z \tan zT - \beta}.$$

Then, $h_1(z)$ satisfies the condition of a theorem on expansion in rational

functions,* and has poles $\pm\beta\rho_j$ and the residues $a(\beta\rho_j)$, $j = 0, 1, 2, \cdots$. Hence, according to the theorem,

$$\sum_{j=0}^{\infty} a(\beta\rho_j)\left(\frac{1}{\alpha\sigma_i + \beta\rho_j} + \frac{1}{\alpha\sigma_i - \beta\rho_j}\right) = h_1(\alpha\sigma_i) = \frac{\alpha\sigma_i}{\alpha - \beta}, \tag{15}$$

where (3) is used for the last equality. Thus,

$$\sum_{j=0}^{\infty} \frac{4\dfrac{\beta}{\alpha}(\alpha - \beta)^2 \cos^2 \beta\rho_j T}{\left(T + \dfrac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i}\right)} \frac{a(\beta\rho_j)}{2\alpha^3\sigma_i^3}\left(\frac{1}{\alpha\sigma_i + \beta\rho_j} + \frac{1}{\alpha\sigma_i - \beta\rho_j}\right) < \frac{2\beta(\alpha - \beta)}{\alpha^3\sigma_0^2 T} \tag{16}$$

where (12) is also used.

Therefore, upon combination of (11), (14), and (16), together with (7), we conclude that for even k's $\sum_l (\nu_l/\mu_k)(f_k, g_l)^2$ is bounded by the sum of the right-hand sides of (11), (14), and (16), which is obviously independent of k.

For odd k's, we can arrive at the same conclusion by following the similar steps.

A.3 *Formal Solutions of* $R_2\psi = \lambda R_1\psi$

The formal solutions of the homogeneous integral equation

$$\int_{-T}^{T} \exp(-\beta|s - t|)\psi_k(s)\, ds = \lambda_k \int_{-T}^{T} \exp(-\alpha|s - t|)\psi_k(s)\, ds$$

are

$$\lambda_{2i} = \frac{\beta}{\alpha}\frac{\alpha^2 + \theta_i^2}{\beta^2 + \theta_i^2}, \qquad \lambda_{2i+1} = \frac{\beta}{\alpha}\frac{\alpha^2 + \hat{\theta}_i^2}{\beta^2 + \hat{\theta}_i^2}, \qquad i = 0, 1, 2, \cdots, \tag{17}$$

$$\begin{aligned} \psi_{2i}(t) &= \cos\theta_i t + \frac{\cos\theta_i T}{\alpha + \beta}[\delta(t - T) + \delta(t + T)], \\ \psi_{2i+1}(t) &= \sin\hat{\theta}_i t + \frac{\sin\hat{\theta}_i T}{\alpha + \beta}[\delta(t - T) - \delta(t + T)], \end{aligned} \tag{18}$$

where θ_i and $\hat{\theta}_i$ are positive solutions of

$$\begin{aligned} (\alpha + \beta)\theta_i \tan\theta_i T &= \alpha\beta - \theta_i^2, \\ -(\alpha + \beta)\hat{\theta}_i \operatorname{ctn}\hat{\theta}_i T &= \alpha\beta - \hat{\theta}_i^2, \end{aligned} \tag{19}$$

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* See Ref. 8, p. 134.

respectively, and they are indexed in ascending order. Namely, if θ_i and $\hat{\theta}_i$, $i = 0, 1, 2, \cdots$, are solutions of (19), then the following equalities hold for every i:

$$\begin{aligned}
&\int_{-T}^{T} \exp(-\beta|s-t|) \cos \theta_i s \, ds \\
&\qquad + \frac{\cos \theta_i T}{\alpha+\beta} \{\exp[-\beta(T-t)] + \exp[-\beta(T+t)]\} \\
&= \frac{\beta}{\alpha} \frac{\alpha^2+\theta_i^2}{\beta^2+\theta_i^2} \left[\int_{-T}^{T} \exp(-\alpha|s-t|) \cos \theta_i s \, ds \right. \\
&\qquad \left. + \frac{\cos \theta_i T}{\alpha+\beta} \{\exp[-\alpha(T-t)] + \exp[-\alpha(T+t)]\} \right],
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\int_{-T}^{T} \exp(-\beta|s-t|) \sin \hat{\theta}_i s \, ds \\
&\qquad + \frac{\sin \hat{\theta}_i T}{\alpha+\beta} \{\exp[-\beta(T-t)] - \exp[-\beta(T+t)]\} \\
&= \frac{\beta}{\alpha} \frac{\alpha^2+\hat{\theta}_i^2}{\beta^2+\hat{\theta}_i^2} \left[\int_{-T}^{T} \exp(-\alpha|s-t|) \sin \hat{\theta}_i s \, ds \right. \\
&\qquad \left. + \frac{\sin \hat{\theta}_i T}{\alpha+\beta} \{\exp[-\alpha(T-t)] - \exp[-\alpha(T+t)]\} \right].
\end{aligned}$$

The above assertion can be verified through direct calculation.

A.4 *Eigenvalues and Eigenfunctions of* $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$

A.4.1 $\{R_1^{\frac{1}{2}} \psi_{kl}\}_l$ *form Cauchy sequences*

Let ψ_{kl}, $k, l = 0, 1, 2, \cdots$, be the lth partial sum of the series obtained by formally expanding ψ_k of (18) relative to $\{f_k\}$ of (2), the eigenfunctions of R_1. Namely, for $i = 0, 1, 2, \cdots$.

$$\begin{aligned}
\psi_{2i,2n}(t) &= \psi_{2i,2n+1}(t), \\
&= \sum_{j=0}^{n} \left[\int_{-T}^{T} \cos \theta_i s \, f_{2j}(s) \, ds + 2 \frac{\cos \theta_i T}{\alpha+\beta} f_{2j}(T) \right] f_{2j}(t),
\end{aligned} \tag{21}$$

$$\begin{aligned}
\psi_{2i+1,2n+1}(t) &= \psi_{2i+1,2n+2}(t) \\
&= \sum_{j=0}^{n} \left[\int_{-T}^{T} \sin \hat{\theta}_i s \, f_{2j+1}(s) \, ds \right. \\
&\qquad \left. + 2 \frac{\sin \hat{\theta}_i T}{\alpha+\beta} f_{2j+1}(T) \right] f_{2j+1}(t).
\end{aligned}$$

Then, $\{R_1^{\frac{1}{2}}\psi_{kl}\}_l$ forms a Cauchy sequence for every k.

Proof: It suffices to show that for every k

$$\lim_{l\to\infty} \| R_1^{\frac{1}{2}}\psi_{kl} \|^2 < \infty.$$

From (1) and (2) and through the use of (19),

$$\| R_1^{\frac{1}{2}}\psi_{2i,2n} \|^2 = \sum_{j=0}^{n} \left[\int_{-T}^{T} \cos \theta_i s\, f_{2j}(s)\, ds + 2\frac{\cos \theta_i T}{\alpha + \beta} f_{2j}(T) \right]^2 \frac{2}{\alpha(1 + \sigma_j^2)}$$

$$= \frac{4\alpha}{(\alpha + \beta)^2} \cos^2 \theta_i T \sum_{j=0}^{n} a(\alpha\sigma_j) \frac{2\alpha^2(1 + \sigma_j^2)}{(\theta_i^2 - \alpha^2\sigma_j^2)^2}.$$

Note also that

$$\frac{2\alpha^2(1 + \sigma_j^2)}{(\theta_i^2 - \alpha^2\sigma_j^2)^2} = 2\frac{\alpha^2 + \theta_i^2}{(\theta_i^2 - \alpha^2\sigma_j^2)^2} - \frac{1}{\theta_i}\left(\frac{1}{\theta_i + 2\sigma_j} + \frac{1}{\theta_i - \alpha\sigma_j}\right).$$

Next, with the aid of (2), (3), and (19), we find

$$\left[\int_{-T}^{T} \cos \theta_i t\, f_{2j}(t)\, dt\right] = \frac{4(\alpha^2 + \theta_i^2)^2 \cos^2 \theta_k T}{(\alpha + \beta)^2} \frac{a(\alpha\sigma_j)}{(\theta_i^2 - \alpha^2\sigma_j^2)^2}.$$

Also observe that

$$\sum_{j=0}^{\infty} \left[\int_{-T}^{T} \cos \theta_i t\, f_{2j}(t)\, dt\right]^2 = \int_{-T}^{T} \cos^2 \theta_i t\, dt = T + \frac{\sin 2\theta_i T}{2\theta_i}.$$

Thus,

$$\frac{4\alpha}{(\alpha + \beta)^2} \cos^2 \theta_i T \sum_{j=0}^{\infty} a(\alpha\sigma_j) \frac{\alpha(\alpha^2 + \theta_i^2)}{(\theta_i^2 - \alpha^2\sigma_j^2)^2} = \frac{2\alpha}{\alpha^2 + \theta_i^2}\left(T + \frac{\sin 2\theta_i T}{2\theta_i}\right).$$

Next, following the procedure for obtaining (15),

$$-\frac{4\alpha}{(\alpha + \beta)^2} \cos^2 \theta_i T \sum_{j=0}^{\infty} \frac{a(\alpha\sigma_j)}{\theta_i}\left(\frac{1}{\theta_i + \alpha\sigma_j} + \frac{1}{\theta_i - \alpha\sigma_j}\right)$$

$$= \frac{4\alpha}{\alpha + \beta} \frac{\cos^2 \theta_i T}{\alpha^2 + \theta_i^2},$$

where (19) is also used. Hence,

$$\lim_{l\to\infty} \| R_1^{\frac{1}{2}}\psi_{2i,l} \|^2 = \frac{2\alpha}{\alpha^2 + \theta_i^2}\left(T + \frac{\sin 2\theta_i T}{2\theta_i} + \frac{2\cos^2 \theta_i T}{\alpha + \beta}\right) = \frac{2\alpha\tau(\theta_i)}{\alpha^2 + \theta_i^2},$$

where

$$\tau(\theta) = T + \frac{(\alpha + \beta)\alpha\beta}{\theta^4 + (\alpha^2 + \beta^2)\theta^2 + \alpha^2\beta^2}, \tag{22}$$

and (19) is used for the second equality.

By following the same steps, we obtain

$$\lim_{l\to\infty} \| R_1^{\frac{1}{2}}\psi_{2i+1,l} \|^2 = \frac{2\alpha\tau(\hat{\theta}_i)}{\alpha^2 + \hat{\theta}_i^2}.$$

Thus, the assertion is proved.

A.4.2 *Orthornormality of* $\{\varphi_k\}$

Define φ_{km}, $k, m = 0, 1, 2, \cdots$, by

$$\begin{aligned} \varphi_{2i,2n} = \varphi_{2i,2n+1} &= \left(\lim_{l\to\infty} \| R_1^{\frac{1}{2}}\psi_{2i,l} \|^2\right)^{-\frac{1}{2}} R_1^{\frac{1}{2}}\psi_{2i,2n}, \\ \varphi_{2i+1,2n+1} = \varphi_{2i+1,2n+2} &= \left(\lim_{l\to\infty} \| R_1^{\frac{1}{2}}\psi_{2i+1,l} \|^2\right)^{-\frac{1}{2}} R_1^{\frac{1}{2}}\psi_{2i+1,2n+1}. \end{aligned} \tag{23}$$

Define φ_k by

$$\varphi_{2i} = \underset{n\to\infty}{\text{l.i.m.}}\ \varphi_{2i,2n}, \qquad \varphi_{2i+1} = \underset{n\to\infty}{\text{l.i.m.}}\ \varphi_{2i+1,2n+1}. \tag{24}$$

Then, $\{\varphi_k\}$ is a sequence of orthonormal functions.

Proof: Normality is obvious. To prove orthogonality, let us first write φ_k explicitly.

$$\begin{aligned} \varphi_{2i}(t) &= \underset{n\to\infty}{\text{l.i.m.}}\ b_i \sum_{j=0}^{n} \frac{[2\alpha^2(1 + \sigma_j^2)]^{\frac{1}{2}} \cos \alpha\sigma_j T}{\left(T + \dfrac{\sin 2\alpha\sigma_j T}{2\alpha\sigma_j}\right)(\alpha^2\sigma_j^2 - \theta_i^2)} \cos \alpha\sigma_j t, \\ \varphi_{2i+1}(t) &= \underset{n\to\infty}{\text{l.i.m.}}\ \hat{b}_i \sum_{j=0}^{n} \frac{[2\alpha^2(1 + \hat{\sigma}_j^2)]^{\frac{1}{2}} \sin \alpha\hat{\sigma}_j T}{\left(T - \dfrac{\sin 2\alpha\hat{\sigma}_j T}{2\alpha\hat{\sigma}_j}\right)(\alpha^2\hat{\sigma}_j^2 - \hat{\theta}_i^2)} \sin \alpha\hat{\sigma}_j t, \end{aligned} \tag{25}$$

where

$$b_i = \left[\frac{2(\alpha^2 + \theta_i^2)}{\tau(\theta_i)}\right]^{\frac{1}{2}} \frac{\cos \theta_i T}{\alpha + \beta}, \qquad \hat{b}_i = \left[\frac{2(\alpha^2 + \hat{\theta}_i^2)}{\tau(\hat{\theta}_i)}\right]^{\frac{1}{2}} \frac{\sin \hat{\theta}_i T}{\alpha + \beta}. \tag{26}$$

First, note

$$(\varphi_{2i}, \varphi_{2m+1}) = 0, \qquad i, m = 0, 1, 2, \cdots.$$

Secondly,

$$(\varphi_{2i}, \varphi_{2m}) = b_i b_m \sum_{j=0}^{\infty} a(\alpha\sigma_j) \frac{2\alpha^2(1 + \sigma_j^2)}{(\theta_i^2 - \alpha^2\sigma_j^2)(\theta_m^2 - \alpha^2\sigma_j^2)},$$

and

$$\frac{2\alpha^2(1+\sigma_j^2)}{(\theta_i^2-\alpha^2\sigma_j^2)(\theta_m^2-\alpha^2\sigma_j^2)} = \frac{1}{\theta_m^2-\theta_i^2}\left[\frac{\alpha^2+\theta_i^2}{\theta_1}\left(\frac{1}{\theta_i+\alpha\sigma_j}+\frac{1}{\theta_i-\alpha\sigma_j}\right)\right.$$
$$\left. -\frac{\alpha^2+\theta_m^2}{\theta_m}\left(\frac{1}{\theta_m+\alpha\sigma_j}+\frac{1}{\theta_m-\alpha\sigma_j}\right)\right].$$

Again, following the same procedure for obtaining (15),

$$(\varphi_{2i},\varphi_{2m}) = \frac{b_ib_m}{\theta_m^2-\theta_i^2}\left(\frac{\alpha^2+\theta_i^2}{\theta_i}\frac{\theta_i}{\theta_i\tan\theta_iT-\alpha}\right.$$
$$\left.-\frac{\alpha^2+\theta_m^2}{\theta_m}\frac{\theta_m}{\theta_m\tan\theta_mT-\alpha}\right) = 0,$$

where (19) is used for the second equality.

By following the same steps, we also obtain

$$(\varphi_{2i+1},\varphi_{2m+1}) = 0.$$

A.4.3 $\{\varphi_k\}$ *forms an orthonormal basis of* $\mathfrak{L}_2$

Since $\{\varphi_k\}$ is a sequence of orthonormal functions and $\{f_l\}$ is an orthonormal basis of $\mathfrak{L}_2$, it suffices to show that for every $l = 0, 1, 2, \cdots$,

$$\sum_{k=0}^{\infty}(f_l,\varphi_k)^2 = 1.$$

First, note that (f_l,φ_k) vanishes unless l and k have the same parity. Secondly, from (2) and (25),

$$\sum_{j=0}^{\infty}(f_{2i},\varphi_{2j})^2 = \frac{1}{(\alpha+\beta)\left(T+\dfrac{\sin 2\alpha\sigma_iT}{2\alpha\sigma_i}\right)}$$
$$\sum_{j=0}^{\infty}\frac{\cos^2\theta_jT}{(\alpha+\beta)\tau(\theta_j)}\frac{4\alpha^2\sigma_i^2(\alpha^2+\theta_j^2)}{(\alpha^2\sigma_i^2-\theta_j^2)^2},$$

and also note

$$\frac{4\alpha^2\sigma_j^2(\alpha^2+\theta_j^2)}{(\alpha^2\sigma_i^2-\theta_j^2)^2} = \alpha^2(1+\sigma_i^2)\left[\frac{1}{(\alpha\sigma_i+\theta_j)^2}+\frac{1}{(\alpha\sigma_i-\theta_j)^2}\right]$$
$$+\frac{\alpha^2(1-\sigma_i^2)}{\alpha\sigma_i}\left(\frac{1}{\alpha\sigma_i+\theta_j}+\frac{1}{\alpha\sigma_i-\theta_j}\right).$$

Now, consider a function of the complex variable defined by

$$h_2(z) = \frac{z}{(\alpha+\beta)z\tan zT-\alpha\beta+z^2}.$$

$h_2(z)$ satisfies the condition of the previously quoted theorem and has poles $\pm\theta_j$ and the residues

$$\frac{\cos^2 \theta_j T}{(\alpha + \beta)\tau(\theta_j)}, \qquad j = 0, 1, 2, \cdots .$$

Also,

$$\frac{d}{dz} h_2(z) \bigg|_{z=\alpha\sigma_i} = \frac{1}{\alpha^2(1 + \sigma_i^2)} \left[\frac{1 - \sigma_i^2}{1 + \sigma_i^2} - (\alpha + \beta) \left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i} \right) \right].$$

Thus, according to the theorem,

$$\frac{\alpha^2(1 - \sigma_i^2)}{\alpha\sigma_i} \sum_{j=0}^{\infty} \frac{\cos^2 \theta_j T}{(\alpha + \beta)\tau(\theta_j)} \left(\frac{1}{\alpha\sigma_i + \theta_j} + \frac{1}{\alpha\sigma_i - \theta_j} \right)$$
$$= \frac{\alpha^2(1 - \sigma_i^2)}{\alpha\sigma_i} h_2(\alpha\sigma_i) = \frac{1 - \sigma_i^2}{1 + \sigma_i^2},$$

where (3) is used for the second equality. Also, through the use of a modified version of the theorem,*

$$\alpha^2(1 + \sigma_i^2) \sum_{j=0}^{\infty} \frac{\cos^2 \theta_j T}{(\alpha + \beta)\tau(\theta_j)} \left[\frac{1}{(\alpha\sigma_i + \theta_j)^2} + \frac{1}{(\alpha\sigma_i - \theta_j)^2} \right]$$
$$= -\alpha^2(1 + \sigma_i^2) \frac{d}{dz} h_2(z) \bigg|_{z=\alpha\sigma_i} = (\alpha + \beta)$$
$$\cdot \left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i} \right) - \frac{1 - \sigma_i^2}{1 + \sigma_i^2}.$$

Hence, upon combination of these two results,

$$\sum_{j=0}^{\infty} \frac{\cos^2 \theta_j T}{(\alpha + \beta)\tau(\theta_j)} \frac{4\alpha^2\sigma_i^2(\alpha^2 + \theta_j^2)}{(\alpha^2\sigma_i^2 - \theta_j^2)^2} = (\alpha + \beta) \left(T + \frac{\sin 2\alpha\sigma_i T}{2\alpha\sigma_i} \right).$$

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* The modified version:
Let $f(z)$ be the function satisfying the condition of the theorem (Ref. 8, p. 134), having poles a_n and their residues b_n. Then,

$$\sum_n \frac{b_n}{(a_n - x)^2} = -\frac{d}{dz} f(z) \bigg|_{z=x}.$$

This is proved by noting that

$$\frac{1}{2\pi i} \int_{C_m} \frac{f(z)}{(z - x)^2} dz = \frac{d}{dz} f(z) \bigg|_{z=x} + \sum_n \frac{b_n}{(a_n - x)^2},$$

and the left-hand side vanishes as $m \to \infty$, where C_m is the contour defined in Ref. 8.

Thus,

$$\sum_{j=0}^{\infty} (f_{2i}, \varphi_{2j})^2 = 1, \qquad i = 0, 1, 2, \cdots.$$

By following similar steps, we obtain

$$\sum_{j=0}^{\infty} (f_{2i+1}, \varphi_{2j+1})^2 = 1, \qquad i = 0, 1, 2, \cdots,$$

and the assertion is proved.

A.4.4 *Closed form-expressions of* $R_1^{\frac{1}{2}}\varphi_k$

$$R_1^{\frac{1}{2}}\varphi_{2i}(t) = \left[\frac{2\alpha}{(\alpha^2+\theta_i^2)\tau(\theta_i)}\right]^{\frac{1}{2}} \cos\theta_i t,$$

$$i = 0, 1, 2, \cdots. \qquad (27)$$

$$R_1^{\frac{1}{2}}\varphi_{2i+1}(t) = \left[\frac{2\alpha}{(\alpha^2+\theta_i^2)\tau(\hat{\theta}_i)}\right]^{\frac{1}{2}} \sin\hat{\theta}_i t,$$

Proof: From (25) and (1),

$$-R_1^{\frac{1}{2}}\varphi_{2i}(t) = \lim_{n\to\infty} \frac{b_i\alpha^{\frac{1}{2}}}{\theta_i} \sum_{j=0}^{n} c_j(t)\left(\frac{1}{\theta_i+\alpha\sigma_j} + \frac{1}{\theta_i-\alpha\sigma_j}\right)$$

$$-R_1^{\frac{1}{2}}\varphi_{2i+1}(t) = \lim_{n\to\infty} \frac{\hat{b}_i\alpha^{\frac{1}{2}}}{\hat{\theta}_i} \sum_{j=0}^{n} \hat{c}_j(t)\left(\frac{1}{\hat{\theta}_i+\alpha\hat{\sigma}_j} + \frac{1}{\hat{\theta}_i-\alpha\hat{\sigma}_j}\right),$$

where b_i and $\hat{b}_i$ are given by (26) and

$$c_j(t) = \frac{\cos\alpha\sigma_j T\cos\alpha\sigma_j t}{T + (\sin 2\alpha\sigma_j T/2\alpha\sigma_j)}, \qquad \hat{c}_j(t) = \frac{\sin\alpha\hat{\sigma}_j T\sin\alpha\hat{\sigma}_j t}{T - (\sin 2\alpha\hat{\sigma}_j T/2\hat{\sigma}_j)}.$$

In order to sum the series, consider

$$h_3(z) = \frac{z\cos zt}{z\sin zT - \alpha\cos zT}.$$

Observe that $h_3(z)$ satisfies the condition of the previously quoted theorem, and has poles $\pm\alpha\sigma_j$ and the residues $c_j(t)$, $j = 0, 1, 2, \cdots$.* Thus, with the use of the theorem,

$$-R_1^{\frac{1}{2}}\varphi_{2i}(t) = \frac{b_i\alpha^{\frac{1}{2}}}{\theta_i} h_3(\theta_i) = -\frac{\alpha^{\frac{1}{2}}(\alpha+\beta)b_i}{(\alpha^2+\theta_i^2)\cos\theta_i T}\cos\theta_i t,$$

* The residues of $h_3(z)$ are shown to be $c_j(t)$ through direct calculation with the use of (3).

where (19) is used for the second equality. Similarly,

$$-R_1^{\frac{1}{2}}\varphi_{2i+1}(t) = -\frac{\alpha^{\frac{1}{2}}(\alpha+\beta)\hat{b}_i}{(\alpha^2+\hat{\theta}_i^2)\sin\hat{\theta}_iT}\sin\hat{\theta}_it.$$

Then, substitution of (26) into the above gives (27).

A.4.5 λ_k *and* φ_k *are eigenvalues and eigenfunctions of* $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$

λ_k and φ_k, $k = 0, 1, 2, \cdots$, are the eigenvalues and the corresponding orthonormalized eigenfunctions of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ to the whole of $\mathfrak{L}_2$.

Proof: It suffices to show that for every k

$$\lambda_kR_1^{\frac{1}{2}}\varphi_k = \operatorname*{l.i.m.}_{m\to\infty} R_2R_1^{-\frac{1}{2}}\varphi_{km},$$

where φ_{km}, $m = 0, 1, 2, \cdots$, are defined by (23), namely, φ_{km} is the mth partial sum of the series obtained by expanding φ_k relative to $\{f_l\}$.

Through direct calculation,

$$\int_{-T}^{T}\exp(-\beta|s-t|)\cos\theta_is\,ds$$

$$+\frac{\cos\theta_iT}{\alpha+\beta}\{\exp[-\beta(T-t)]+\exp[-\beta(T+t)]\} = \frac{2\beta}{\beta^2+\theta_i^2}\cos\theta_it.$$

Thus, from (17) and (27),

$$\lambda_{2i}(R_1^{\frac{1}{2}}\varphi_{2i})(t) = \frac{(\alpha+\beta)b_i}{2\alpha^{\frac{1}{2}}\cos\theta_iT}\left[\int_{-T}^{T}\exp(-\beta|s-t|)\cos\theta_is\,ds\right.$$
$$\left.+\frac{\cos\theta_iT}{\alpha+\beta}\{\exp[-\beta(T-t)]+\exp[-\beta(T+t)]\}\right].$$

On the other hand, from (23) and (21),

$$(R_2R_1^{-\frac{1}{2}}\varphi_{2i,2n})(t) = (R_2R_1^{-\frac{1}{2}}\varphi_{2i,2n+1})(t)$$
$$= \frac{(\alpha+\beta)b_i}{2\alpha^{\frac{1}{2}}\cos\theta_iT}\sum_{j=0}^{n}(R_2f_{2j})(t)\left[\int_{-T}^{T}\cos\theta_is\,f_{2j}(s)\,ds\right.$$
$$\left.+2\frac{\cos\theta_iT}{\alpha+\beta}f_{2j}(T)\right].$$

But, since

$$\int_{-T}^{T}\exp(-\beta|s-t|)\cos\theta_is\,ds = \operatorname*{l.i.m.}_{n\to\infty}\sum_{j=0}^{n}(R_2f_{2j})(t)\int_{-T}^{T}\cos\theta_is\,f_{2j}(s)\,ds,$$

we only have to show

$$\exp[-\beta(T-t)] + \exp[-\beta(T+t)] = 2 \lim_{n\to\infty} \sum_{j=0}^{n} f_{2j}(T)(R_2 f_{2j})(t),$$

i.e.,

$$R_2(T,t) + R_2(-T,t) = \operatorname*{l.i.m.}_{n\to\infty} \sum_{l=0}^{n} [f_l(T) + f_l(-T)](R_2 f_l)(t).$$

But this is certainly implied by

$$R_2^2(s,t) = \sum_{k,l=0}^{\infty} (f_k, R_2^2 f_l) f_k(s) f_l(t), \qquad -T \leqq s,t \leqq T, \tag{28}$$

where

$$R_2^2(s,t) = \int_{-T}^{T} R_2(s,u) R_2(u,t)\, du.$$

To prove (28), we first note that the series on the right converges to $R_2^2(s,t)$ in the mean. In addition,

$$| f_k(t) | < T^{-\frac{1}{2}},$$

as seen from (2) and (12). Hence, it suffices to show that

$$\sum_{k,l=0}^{\infty} | (f_k, R_2^2 f_l) | < \infty,$$

which is implied, through the Schwarz inequality, by

$$\sum_{k=0}^{\infty} \| R_2 f_k \|^2 < \infty.$$

Hence, we have shown that

$$\lambda_{2i} R_1^{\frac{1}{2}} \varphi_{2i} = \lim_{m\to\infty} R_2 R_1^{-\frac{1}{2}} \varphi_{2i,m}, \qquad i = 0, 1, 2, \cdots.$$

Through the same argument, (28) implies

$$\lambda_{2i+1} R_1^{\frac{1}{2}} \varphi_{2i+1} = \lim_{m\to\infty} R_2 R_1^{-\frac{1}{2}} \varphi_{2i+1,m}.$$

APPENDIX B

Example 2

$$\left.\begin{aligned} R_1(s,t) &= 1 - \frac{|s-t|}{2T}, \\ R_2(s,t) &= \exp\left(-\frac{|s-t|}{T}\right), \end{aligned}\right\} \quad -T \leqq s,t \leqq T.$$

B.1 *Eigenvalues and Eigenfunctions of R_1 and R_2*

The eigenvalues and the orthonormalized eigenfunctions of R_1 are

$$\mu_{2i} = \frac{1}{T\theta_i^2}, \qquad \mu_{2i+1} = \frac{T}{(i + \frac{1}{2})^2\pi^2}, \qquad i = 0, 1, 2, \cdots, \tag{29}$$

$$f_{2i}(t) = \frac{\cos \theta_i t}{\left(T + \dfrac{\sin 2\theta_i T}{2\theta_i}\right)^{\frac{1}{2}}}, \qquad f_{2i+1}(t) = \frac{\sin \left(i + \dfrac{1}{2}\right) \dfrac{\pi}{T} t}{T^{\frac{1}{2}}}, \tag{30}$$

where θ_i, $i = 0, 1, 2, \cdots$, are positive solutions of

$$\theta_i T \tan \theta_i T = 1, \tag{31}$$

indexed in ascending order. Similarly, the eigenvalues and eigenfunctions of R_2 are

$$\nu_{2i} = \frac{2T}{1 + \theta_i^2 T^2}, \qquad \nu_{2i+1} = \frac{2T}{1 + \hat{\theta}_i^2 T^2}, \tag{32}$$

$$g_{2i}(t) = f_{2i}(t), \qquad g_{2i+1}(t) = \frac{\sin \hat{\theta}_i t}{\left(T - \dfrac{\sin 2\hat{\theta}_i T}{2\hat{\theta}_i}\right)^{\frac{1}{2}}}, \tag{33}$$

where $\hat{\theta}_i$, $i = 0, 1, 2, \cdots$, are positive solutions of

$$-\hat{\theta}_i T \operatorname{ctn} \hat{\theta}_i T = 1. \tag{34}$$

B.2 *Boundedness of $R_2^{\frac{1}{2}} R_1^{-\frac{1}{2}}$*

From (29), (32), and (33),

$$\| R_2^{\frac{1}{2}} R_1^{-\frac{1}{2}} f_{2i} \|^2 = \frac{2\theta_i^2 T^2}{1 + \theta_i^2 T^2} < 2, \qquad i = 0, 1, 2, \cdots.$$

Since

$$(f_{2i+1}, g_{2j}) = 0, \qquad j = 0, 1, 2, \cdots,$$

we have, through (29), (30), (32), and (33),

$$\| R_2^{\frac{1}{2}} R_1^{-\frac{1}{2}} f_{2i+1} \|^2 = \sum_{j=0}^{\infty} \frac{2\hat{\theta}_j^2 T^2}{1 + \hat{\theta}_j^2 T^2} \, \frac{4T \cos^2 \hat{\theta}_j T}{T - \dfrac{\sin 2\hat{\theta}_j T}{2\hat{\theta}_j}} \, \frac{(i + \frac{1}{2})^2 \pi^2}{[\hat{\theta}_j^2 T^2 - (i + \frac{1}{2})^2 \pi^2]^2},$$

and also note

$$\frac{(i+\frac{1}{2})^2\pi^2}{[\hat{\theta}_j^2T^2-(i+\frac{1}{2})^2\pi^2]^2}=\frac{\hat{\theta}_j^2T^2}{[\hat{\theta}_j^2T^2-(i+\frac{1}{2})^2\pi^2]^2}-\frac{1}{\theta_j^2T^2-(i+\frac{1}{2})^2\pi^2}. \tag{35}$$

Now

$$\sum_{j=0}^{\infty}\frac{4T\cos^2\hat{\theta}_jT}{T-\dfrac{\sin 2\hat{\theta}_jT}{2\hat{\theta}_j}}\frac{\hat{\theta}_j^2T^2}{[\hat{\theta}_jT^2-(i+\frac{1}{2})^2\pi^2]^2}=\sum_{j=0}^{\infty}(f_{2i+1},g_{2j+1})^2=1, \tag{36}$$

$$\left|\sum_{j=0}^{\infty}\frac{2\hat{\theta}_iT^2}{1+\hat{\theta}_j^2T^2}\frac{T\cos^2\hat{\theta}_jT}{T-\dfrac{\sin 2\hat{\theta}_jT}{2\hat{\theta}_j}}\frac{1}{\hat{\theta}_j^2T^2-(i+\frac{1}{2})^2\pi^2}\right|$$

$$\leqq\sum_{j=0}^{\infty}\frac{\hat{\theta}_j^2T^2}{(1+\hat{\theta}_j^2T^2)^2}\frac{T\cos^2\hat{\theta}_jT}{T-\dfrac{\sin 2\hat{\theta}_jT}{2\hat{\theta}_j}}<\infty,$$

where the first inequality follows from the Schwarz inequality and (36) while the second follows from (34).* Hence, $\| R_2^{\frac{1}{2}}R_1^{-\frac{1}{2}}f_{2i+1}\|^2$ is also bounded by a constant independent of i.

Thus, by using the argument in A.2, we conclude that $R_2^{\frac{1}{2}}R_1^{-\frac{1}{2}}$ is bounded.

B.3 *Formal Solutions of* $R_1\psi=\lambda R_0\psi$

Unlike Example 1, the even solutions of

$$\int_{-T}^{T}\exp\left(-\frac{|s-t|}{T}\right)\psi_k(s)\,ds=\lambda_k\int_{-T}^{T}\left(1-\frac{|s-t|}{2T}\right)\psi_k(s)\,ds \tag{37}$$

are bonafide functions while the odd remain formal. They are

$$\lambda_{2i}=\lambda_{2i+1}=\frac{2\theta^2T^2}{1+\theta_i^2T^2}, \tag{38}$$

$$\begin{aligned}\psi_{2i}(t)&=\cos\theta_it,\\ \psi_{2i+1}(t)&=\sin\theta_it+T\sin\theta_iT[\delta(t-T)-\delta(t+T)].\end{aligned} \tag{39}$$

* It follows from (34) that

$$T-\frac{\sin 2\hat{\theta}_jT}{2\hat{\theta}_j}=T\left(1+\frac{1}{1+\hat{\theta}_j^2T^2}\right)>T,$$

which corresponds to (12).

Again, the precise meaning of the odd solutions is that if θ_i, $i = 0, 1, 2, \cdots$, are positive solutions of (31) then

$$\int_{-T}^{T} \exp\left(-\frac{|s-t|}{T}\right) \sin \theta_i s \, ds + T \sin \theta_i T \cdot \left[\exp\left(-\frac{T-t}{T}\right) - \exp\left(-\frac{T+t}{T}\right)\right] = \frac{2\theta_i^2 T^2}{1+\theta_i^2 T^2} \int_{-T}^{T} \left(1 - \frac{|s-t|}{2T}\right) \sin \theta_i s \, ds + t \sin \theta_i T \tag{40}$$

for every i. Again, the above assertions can be verified through direct calculation.

B.4 *Eigenvalues and Eigenfunctions of* $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$

B.4.1 $\{R_1^{\frac{1}{2}} \psi_{2i+1,l}\}_l$ *form Cauchy sequences**

Define, for each $i = 0, 1, 2, \cdots$,

$$\begin{aligned} \psi_{2i+1,2n+1}(t) &= \psi_{2i+1,2n+2}(t) \\ &= \sum_{j=0}^{n} \left[\int_{-T}^{T} \sin \theta_i s \, f_{2j+1}(s) \, ds + 2T \sin \theta_i T \, f_{2j+1}(T)\right] f_{2j+1}(t). \end{aligned} \tag{41}$$

Then, $\{R_1^{\frac{1}{2}} \psi_{2i+1,l}\}_l$ forms a Cauchy sequence for every i.

Proof: From (29) and (30) and through the use of (31) and (35)

$$\begin{aligned} \| R_1^{\frac{1}{2}} \psi_{2i+1,2n+1} \|^2 &= \sum_{j=0}^{n} \left[\int_{-T}^{T} \sin \theta_i s \, f_{2j+1}(s) \, ds + 2T \sin \theta_i T \, f_{2j+1}(T)\right] \frac{T}{\left(j+\frac{1}{2}\right)^2 \pi^2} \\ &= 4 \, \frac{\cos^2 \theta_i T}{\theta_i^2} \sum_{j=0}^{n} \left(\frac{\theta_i^2 T^2}{\left[\theta_i^2 T^2 - \left(j+\frac{1}{2}\right)^2 \pi^2\right]^2} - \frac{1}{\theta_i^2 T^2 - \left(j+\frac{1}{2}\right)^2 \pi^2}\right). \end{aligned}$$

Now

$$\sum_{j=0}^{n} \frac{4\theta_i^2 T^3 \cos^2 \theta_i T}{\left[\theta_i T^2 - \left(j+\frac{1}{2}\right)^2 \pi^2\right]^2} = \int_{-T}^{T} \sin^2 \theta_i t \, dt = T - \frac{\sin 2\theta_i T}{2\theta_i}. \tag{42}$$

---

* Note there is no need for considering such sequences for the even solutions ψ_{2i}'s, since they are already $\mathfrak{L}_2$-functions.

In order to sum the second term, observe that the function of complex variable $\tan z$ satisfies the condition of the theorem repeatedly used, and has poles $(j + \frac{1}{2})\pi$, $j = 0, \pm 1, \pm 2, \cdots$, and the residues -1. Hence, using the theorem,

$$-\sum_{j=0}^{\infty} \frac{1}{\theta_i^2 T^2 - \left(j + \frac{1}{2}\right)^2 \pi^2} = \frac{\tan \theta_i T}{2\theta_i T}. \tag{43}$$

Thus, combining the two results,

$$\lim_{n\to\infty} \| R_1^{\frac{1}{2}} \psi_{2i+1,2n+1} \|^2 = \frac{1}{\theta_i^2 T}\left(T + \frac{\sin 2\theta_i T}{2\theta_i}\right). \tag{44}$$

Hence, through the use of the argument in A.4.1, the assertion is proved.

B.4.2 *Orthonormality of* $\{\varphi_k\}$

Define φ_k, $k = 0, 1, 2, \cdots$, by

$$\varphi_{2i} = f_{2i}, \qquad \varphi_{2i+1} = \lim_{l\to\infty} \varphi_{2i+1,l}, \tag{45}$$

where $i = 0, 1, 2, \cdots$, and

$$\varphi_{2i+1,\ell}(t) = \left(\lim_{\ell\to\infty} \| R_1 \psi_{2i+1,\ell} \|^2\right)^{-\frac{1}{2}} R_1^{\frac{1}{2}} \psi_{2i+1,\ell}.$$

Then, $\{\varphi_k\}$ is a sequence of orthonormal functions.

Proof: Normality is self-evident. Note, from (30) and (41),

$$(\varphi_{2i}, \varphi_{2m}) = \delta_{im}, \qquad (\varphi_{2i}, \varphi_{2m+1}) = 0, \qquad m = 0, 1, 2, \cdots,$$

and

$$\begin{aligned}
(\varphi_{2i+1}, \varphi_{2m+1}) &= 4Tf_{2i}(T)f_{2m}(T) \\
&\quad \cdot \sum_{j=0}^{\infty} \frac{\left(j + \frac{1}{2}\right)^2 \pi^2}{\left[\theta_i^2 T^2 - \left(j + \frac{1}{2}\right)^2 \pi^2\right]\left[\theta_m^2 T^2 - \left(j + \frac{1}{2}\right)^2 \pi^2\right]} \\
&= \frac{2f_{2i}(T)f_{2m}(T)}{\theta_m^2 - \theta_i^2} \sum_{j=0}^{\infty} \left[\frac{\theta_i}{\theta_i T + \left(j + \frac{1}{2}\right)\pi}\right. \\
&\quad + \frac{\theta_i}{\theta_i T - \left(j + \frac{1}{2}\right)\pi} - \frac{\theta_m}{\theta_m T + \left(j + \frac{1}{2}\right)\pi} \\
&\quad \left. - \frac{\theta_m}{\theta_m T - \left(j + \frac{1}{2}\right)\pi}\right].
\end{aligned}$$

Thus, from (43) and (31),

$$(\varphi_{2i+1}, \varphi_{2m+1}) = 2f_{2i}(T)f_{2m}(T) \frac{\theta_i T \tan \theta_i T - \theta_m T \tan \theta_m T}{\theta_m{}^2 - \theta_i{}^2} = 0$$

B.4.3 *$\{\varphi_k\}$ forms an orthonormal basis of $\mathfrak{L}_2$*

First, note from (30) and (45),

$$\sum_{k=0}^{\infty} (f_{2i}, \varphi_k)^2 = \sum_{j=0}^{\infty} (f_{2i}, f_{2j})^2 = 1, \qquad i = 0, 1, 2, \cdots .$$

Next,

$$\sum_{k=0}^{\infty} (f_{2i+1}, \varphi_k)^2 = \sum_{j=0}^{\infty} (f_{2i+1}, \varphi_{2j+1})^2$$

$$= \sum_{j=0}^{\infty} \frac{4T \cos^2 \theta_j T}{T + \dfrac{\sin 2\theta_j T}{2\theta_j}} \frac{\left(i + \frac{1}{2}\right)^2 \pi^2}{\left[\theta_j{}^2 T^2 - \left(1 + \frac{1}{2}\right)^2 \pi^2\right]^2}$$

$$= \frac{1}{T} \sum_{k=0}^{\infty} \left[\int_{-T}^{T} \cos\left(i + \frac{1}{2}\right) \frac{\pi}{T} s\, f_k(s)\, ds\right]^2$$

$$= 1.$$

Thus, using the argument in A.4.3, the assertion is proved.

B.4.4 *Closed form-expressions of $R_1^{\frac{1}{2}}\varphi_k$*

$$(R_1^{\frac{1}{2}}\varphi_{2i})(t) = d_i \cos \theta_i t, \qquad (R_1^{\frac{1}{2}}\varphi_{2i+1})(t) = d_i \sin \theta_i t, \tag{46}$$

where

$$d_i = \frac{1}{\theta_i T}\left(1 + \frac{1}{1 + \theta_i{}^2 T^2}\right)^{-\frac{1}{2}}.$$

Proof: The even part of (46) follows immediately from the even parts of (45) and (29). Now, from the odd parts of (45) and (29),*

$$-(R_1^{\frac{1}{2}}\varphi_{2i+1})(t) = \operatorname*{l.i.m.}_{n\to\infty} \sum_{j=0}^{n} \frac{2(-1)^j d_i \theta_i T \cos \theta_i T \sin\left(j + \frac{1}{2}\right) \frac{\pi}{T} t}{\theta_i{}^2 T^2 - \left(j + \frac{1}{2}\right)^2 \pi^2}.$$

---

* Note, from (34) that

$$T + \frac{\sin 2\theta_i T}{2\theta_i} = T\left(1 + \frac{1}{1 + \theta_i{}^2 T^2}\right).$$

In order to sum the series, observe that a function of complex variable $\sin z(t/T)/\cos z$ satisfies the condition of the theorem repeatedly used, and has poles $(j + \frac{1}{2})\pi$, $j = 0, \pm 1, \pm 2, \cdots$, and the residues

$$-(-1)^j \sin (j + \tfrac{1}{2})(\pi/T)t.$$

Thus, according to the theorem,

$$\sum_{j=0}^{\infty} \frac{2(-1)^j \sin\left(j + \frac{1}{2}\right)\frac{\pi}{T}t}{\theta_i^2 T^2 - \left(j + \frac{1}{2}\right)^2 \pi^2} = -\frac{\sin \theta_i t}{\theta_i T \cos \theta_i T}.$$

Hence, substitution of the above yields the odd part of (46).

B.4.5 λ_k *and* φ_k *are eigenvalues and eigenfunctions of* $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$

The assertion of A.4.5 holds.

Proof: That λ_{2i} and φ_{2i}, $i = 0, 1, 2, \cdots$, are eigenvalues and eigenfunctions of the extension of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ is easily seen from (29), (32), (33), (38), and (45).*

For the odd part, i.e., λ_{2i+1} and φ_{2i+1}, we need to show that

$$\lambda_{2i+1}R_1^{\frac{1}{2}}\varphi_{2i+1} = \underset{m\to\infty}{\text{l.i.m.}}\ R_2R_1^{-\frac{1}{2}}\varphi_{2i+1,m}. \tag{47}$$

Through direct calculation,

$$\int_{-T}^{T} \exp\left(-\frac{|s-t|}{T}\right)\sin \theta_i s\, ds + T\sin\theta_i T\left[\exp\left(-\frac{T-t}{T}\right) - \exp\left(-\frac{T+t}{T}\right)\right] = \frac{2T}{1+\theta_i^2T^2}\sin\theta_i t.$$

Hence, from (37) and (46),

$$\lambda_{2i+1}(R_1^{\frac{1}{2}}\varphi_{2i+1})(t) = d_i\theta_i^2T^2\left[\int_{-T}^{T}\exp\left(-\frac{|s-t|}{T}\right)\sin\theta_i s\, ds + T\sin\theta_i T\left[\exp\left(-\frac{T-t}{T}\right) - \exp\left(-\frac{T+t}{T}\right)\right]\right].$$

On the other hand, from (45) and (41),

* Note φ_{2i} is an eigenfunction of $R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}}$ itself, without extension to the whole of $\mathfrak{L}_2$.

$$(R_2R_1^{-\frac{1}{2}}\varphi_{2i+1,2n+1})(t) = (R_2R_1^{-\frac{1}{2}}\varphi_{2i+1,2n+2})(t)$$
$$= d_i\theta_i^2T^2 \sum_{j=0}^{n} (R_2f_{2j+1})(t) \left[\int_{-T}^{T} \sin\theta_i s\, f_{2j+1}(s)\, ds + 2T \sin\theta_i T\, f_{2j+1}(T)\right].$$

But, since

$$\int_{-T}^{T} \exp\left(-\frac{|s-t|}{T}\right) \sin\theta_i s\, ds = \underset{n\to\infty}{\mathrm{l.i.m.}} \sum_{j=0}^{n} (R_2f_{2j+1})(t) \cdot \int_{-T}^{T} \sin\theta_i s\, f_{2j+1}(s)\, ds,$$

we only have to show

$$\exp\left(-\frac{T-t}{T}\right) - \exp\left(-\frac{T+t}{T}\right) = 2\lim_{n\to\infty} \sum_{j=0}^{n} f_{2j+1}(T)(R_2f_{2j+1})(t),$$

which is implied by (28).

Then, by following the argument after (28), (47) is proved.*

The second assertion of A.4.5 is valid in this case also, since $\{\varphi_k\}$ of this example forms an orthonormal basis.

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* One non-obvious fact needed here is that $\theta_jT > \pi j$, $j = 1, 2, \cdots$, which follows from (31).